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EFFECT OF THE SMALL FORCED OSCILLATIONS ON THE STABILITY OF THE STEADY MOTION OF FLUID*

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An approximate method is presented of computing the critical values of the problem of stability of viscous incompressible fluid flow in the case when the flow can be represented by superposition of a steady flow and a small, additional time-periodic flow. Usually the steady component is used as the measure of stability of such flows. The development of the method in which the periodic component is taken into account, and the computation of the resonance frequencies possible in such a case, are of interest /1,2/. It is found that in the case which is most interesting from the practical point of view, of harmonic periodic additions to the steady velocity field, the increment in the critical values are equal to zero in the first order approximation with respect to the small parameter, and second order approximations must therefore be obtained. The increments are obtained in explicit form and the possible resonance frequencies shown.

A viscous incompressible fluid moves in three-dimensional space Ω with a boundary of class C^2. Searching for an arbitrary flow $\mathbf{v} = \mathbf{v}_0 + \varepsilon \mathbf{v}_1(t) + \mathbf{u}(t)$ ($\mathbf{v}_0$ is the steady velocity field, ε is a small parameter and $\mathbf{v}_1(t)$ is a time-periodic vecotr function), we arrive at the following equation for the perturbations /3/:

$$d\mathbf{u}/dt = A_0\mathbf{u} + \nu K\mathbf{u} + \varepsilon F(t)\,\mathbf{u} + w(\mathbf{u}, \mathbf{u}) \tag{1}$$
$$A_0\mathbf{u} = -P\,[(\mathbf{u}\nabla)\,\mathbf{v}_0 + (\mathbf{v}_0\nabla)\,\mathbf{u}],\quad K\mathbf{u} = P\Delta\mathbf{u}$$
$$F(t)\,\mathbf{u} = -P\,[(\mathbf{u}\nabla)\,\mathbf{v}_1(t) + (\mathbf{v}_1(t)\,\nabla)\,\mathbf{u}]$$

Here P is the orthogonal projection operator of the vector space $\mathbf{L}_2$ onto the space $\mathbf{H}$ obtained by the closure of the set finite, in Ω, of the smooth solenoidal vectors, with $P(\mathrm{grad}) = 0$ taken into account. The Cauchy operator of the linearized equation (1)

$$d\mathbf{u}/dt = A_0\mathbf{u} + \nu K\mathbf{u} + \varepsilon F(t)\,\mathbf{u} \tag{2}$$

with time-periodic operator can be written with help of the Floquet representation in the form $U(t) = Q(t)\,V(t)$, where $Q(t)$ is a periodic, differentiable operator-function and $V(t)$ is a strongly continuous subgroup of operators with the generating operator N. We seek the critical values ν_* from the equation

$$R\psi = \lambda\psi \tag{3}$$

where $\mathrm{Re}\,\lambda = 0$ /3/ is the condition of criticality, by expanding $Q(t), N, \nu_*, \lambda$ into series in powers of ε

$$N = N_0 + \varepsilon N_1 + \ldots,\quad Q(t) = Q_0(t) + \varepsilon Q_1(t) + \ldots \tag{4}$$
$$\nu_* = \nu_0 + \varepsilon \nu_1 + \ldots,\quad \lambda = \lambda_0 + \varepsilon\lambda_1 + \ldots$$

When $\varepsilon = 0$, we have

$$N = N_0 = A_0 + \nu K,\quad Q_0 = I \tag{5}$$

The operator-function $Q(t, \varepsilon)$ satisfies the equation

$$dQ/dt = AQ - QA,\quad A = A_0 + \nu K + \varepsilon F(t) \tag{6}$$

Substituting into (6) the first three expansions of (4) and separating terms of different order in ε, we obtain the following system of relations

$$dQ_n/dt = N_0 Q_n - Q_n N_0 + C_n(t) - N_n \quad (n = 1, 2, \ldots) \tag{7}$$
$$C_n(t) = G_n(t) + \sum_{j=1}^{n-1} G_j(t)\,Q_{n-j}(t) - \sum_{j=1}^{n-1} Q_j(t)\,N_{n-j}$$
$$C_0 = A_0 + \nu_0 K,\quad G_1 = \nu_1 K + F(t),\quad G_2 = \nu_2 K, \ldots, G_n(t) = \nu_n K$$

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Integrating (7) from zero to T (T is the period of oscillation) and taking into account the equation

$$\int_0^T \frac{dQ}{dt}\, dt = 0$$

we obtain the operator

$$N_n = \frac{1}{T}\int_0^T [N_0 Q_n(t) - Q_n(t) N_0 + C_n(t)]\, dt \quad (8)$$

Expanding now (3) in ε, we obtain

$$N_0\psi_0 = \lambda_0\psi_0, \quad N_1\psi_0 + N_0\psi_1 = \lambda_0\psi_1 + \lambda_1\psi_0 \quad (9)$$

and from the first equation of (9) we find ν_0 and ψ_0.

Let us also consider the following equation conjugate to the first equation of (9):

$$N_0^+\varphi_0 = \bar{\lambda}_0\varphi_0 \quad (10)$$

where N_0^+ is an operator conjugate to N_0. Substituting ν_0 and ψ_0 from the first equation of (9) into the second equation and scalar multiplying in the space H by the vector φ_0, which represents the solution of (10), we obtain $(\varphi_0, N_1, \psi_0) = \lambda_1(\varphi_0, \psi_0)$. Substituting now N_1 from (8) and using the condition of criticality $\mathrm{Re}\,\lambda_1 = 0$, we obtain (with the normalizing factor $(\varphi_0, \psi_0) = 1$)

$$\nu_1 = -\mathrm{Re}\left(\varphi_0, \frac{1}{T}\int_0^T F(t)\, dt\psi_0\right) \Big/ \mathrm{Re}\,(\varphi_0, K\psi_0) \quad (11)$$

From (11) we see that when

$$\int_0^T F(t)\, dt = 0 \quad (12)$$

then the increment in the critical value ν_0 (computed in the steady approximation) of the first order in ε is absent (in this case we obviously also have $\lambda_1 = 0$). Consequently when the condition (12) holds, we must determine the increments to the critical value of higher order in ε. Condition (12) holds, in particular, in the case of harmonic nonstability. In this case it is possible to take into account the terms of second order of smallness in the critical values, and neglect them in the basic flow.

Let the periodic operator-functions $C_n(t)$ and $Q_n(t)$ have the corresponding expansions into the Fourier series

$$C_n(t) = \sum_{p=-\infty}^{\infty} \Phi_{np} \exp\frac{2\pi p i}{T} t, \quad Q_n(t) = \sum_{p=-\infty}^{\infty} D_{np} \exp\frac{2\pi p i}{T} t$$

$$\Phi_{np} = \frac{1}{T}\int_0^T C_n(t) \exp\left(-\frac{2\pi i p}{T} t\right) dt$$

$$D_{np} = \frac{1}{T}\int_0^T Q_n(t) \exp\left(-\frac{2\pi p i}{T} t\right) dt$$

Multiplying the equation (7) by

$$\exp\left(-\frac{2\pi p i}{T} t\right)$$

and integrating with respect to t from zero to T, we obtain the following equation for the operator D_{np} $(p \neq 0)$ /4/:

$$\frac{2\pi p i}{T} D_{np} - N_0 D_{np} + D_{np} N_0 = \Phi_{np} \quad (p \neq 0) \quad (13)$$

For the values of T for which the condition

$$2\pi p/T = i(\mu - \lambda) \quad (\lambda, \mu \in \sigma(N_0))$$

holds and where $\sigma(N_0)$ is the spectrum of the operator N_0, equation (13) has a unique solution /4/

$$D_{np} = -\frac{1}{4\pi^2}\oint_{\gamma_0}\oint_{\gamma_0}\frac{(N_0-\lambda I)^{-1}\Phi_{np}(N_0-\mu I)^{-1}}{2\pi p\iota/T-\lambda+\mu}d\lambda\, d\mu \qquad (14)$$

where γ_0 is the contour encircling $\sigma(N_0)$ sufficiently narrowly.

From the relations $Q_n(0)=\Sigma D_{np}=0$ we obtain $D_{n0}=-\Sigma D_{np}\ (p\neq 0)$ and this, together with (8), yields the following expression for N_n /4/:

$$N_n = N_0D_{n0}-D_{n0}N_0+\Phi_{n0} \qquad (15)$$

Let the condition (12) holds. Then $\Phi_{10}=0$, and we have

$$N_1 = N_0D_{10}-D_{10}N_0 \qquad (16)$$

Let us scalar multiply the equation

$$N_0\psi_2+N_1\psi_1+N_2\psi_0=\lambda_0\psi_2+\lambda_1\psi_1+\lambda_2\psi_0$$

by φ_0 in H. Using the second equation of (9), (4) and (5), we obtain

$$\nu_2(\varphi_0, K\psi_0)+\frac{1}{T}\sum_{p=-\infty}^{\infty}\left(\varphi_0, \int_0^T F(t)\exp\left(\frac{2\pi p\iota}{T}t\right)\times D_{1p}\psi_0 dt\right)=\lambda_2(\varphi_0,\psi_0)$$

Choosing the normalizing $(\varphi_0,\psi_0)=1$ and using the condition of criticality $\operatorname{Re}\lambda_2=0$, we find the following representation for ν_2:

$$\nu_2=-\frac{1}{T\operatorname{Re}(\varphi_0,K\psi_0)}\operatorname{Re}\sum_{p=-\infty}^{\infty}\left(\varphi_0,\int_0^T F(t)\exp\frac{2\pi p\iota t}{T}D_{1p}\psi_0\,dt\right) \qquad (17)$$

It remains to find $D_{1p}\psi_0$ at $p\neq 0$ (since when $p=0$, then the term in the sum (17) vanishes by virtue of (12)). Taking into account the fact that

$$(N_0-\mu I)^{-1}\psi_0=\psi_0/(\lambda-\mu)$$

and using (14), we obtain

$$\nu_2=-\operatorname{Re}\sum_{p=-\infty}^{\infty}(\varphi_0, f_{-p}[N_0-\iota(p\omega+\omega_0)]^{-1}f_p\psi_0)/\operatorname{Re}(\varphi_0,K\psi_0)$$

$$f_p=\frac{1}{T}\int_0^T F(t)\exp\frac{2\pi p\iota}{T}t\,dt,\quad \omega=\frac{2\pi}{T},\quad \omega_0=\frac{\lambda_0}{\iota}$$

i.e. the increment to the critical value $\nu_* = \nu_0+\varepsilon^2\nu_2$. If the imaginary axis in the critical spectrum of the operator N_0 contains, in addition to the point $\lambda_0=\iota\omega_0$ also the point $\lambda'=\iota\omega'$, then the frequencies $\omega=|\omega'-\omega_0|/p$ can become resonant at some p. We must remember here that the spectrum of the operator N_0 is discrete /5/. In connection with the appearance of a resonance it is interesting to recall that the presence in a critical spectrum of operator N_0 of a pair of complex-conjugate points leads to excitation of the self-oscillations /6/. In this case the resonance frequencies are $\omega=2\omega_0/p$.

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